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‘Stable’ density stratification as a catalyst for instability

By D. J. ACHESON
Jesus College, Oxford

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A physical explanation is suggested for how the instability of certain fluid systems may be provoked by the addition of a ‘bottom-heavy’ density gradient. It is shown that in all the recent examples of this behaviour the stratification shifts the oscillation frequency at the marginal state towards the diffusion rate associated with the driving mechanism for the instability, and this allows a more effective release of the available energy. When the driving mechanism is an adverse temperature gradient, for example, the frequency shift induced by a bottom-heavy solute distribution can increase the temperature change that a vertically-displaced fluid parcel acquires during each half-cycle, thereby enhancing the thermal buoyancy forces which drive the instability.

1. Broad outline of the proposed mechanism

Recent theoretical studies by Pearlstein (1977, 1979), Masuda (1978), Roberts (1978), Roberts & Loper (1979), Acheson (1978*a*), Soward (1979) and Fearn (1979) have revealed a number of fluid systems which may apparently be rendered unstable by the addition of a suitably strong ‘bottom-heavy’ density gradient. In this paper we suggest a physical explanation for this curious behaviour.

Consider first the well-known explanation for the thermal over-stability of a fluid layer under gravity when it is heated from below but subject also to a restoring force of some kind. This restoring force is presumed sufficiently large that a displaced fluid parcel would, in the absence of diffusive processes, oscillate about its equilibrium level with constant amplitude. With weak thermal diffusion present, however, the temperature of the parcel continually attempts to adjust towards that of its latest surroundings. Thus, with an adverse basic temperature gradient, the parcel heats up a little during a downward half-cycle and arrives back at the equilibrium level rather lighter than it was at the beginning of that half-cycle. Buoyancy forces then make it overshoot by an amount greater than the previous downward displacement, whereupon the same process (with cooling, instead of heating, of course, above the equilibrium level) continues and leads to a slow growth in amplitude of the oscillations.

This explanation holds good for a variety of different restoring mechanisms, such as the Coriolis force associated with rotation of the reference frame (Chandrasekhar 1961, chap. 2), magnetohydrodynamic (Lorentz) forces (Chandrasekhar 1961, chap. 3), or buoyancy forces stemming from chemical (e.g. salt) stratification of the fluid (Turner 1974). Furthermore, an adverse temperature gradient and thermal diffusion form only one of many combinations by which such over-stability can be driven. In § 2 some of the instabilities discussed are driven by the combination of a magnetic field gradient and ohmic diffusion, but the explanation is essentially the same: the agency which forces a parcel to depart further from its equilibrium position is being more seriously eroded all the time by diffusion than the agency which forces it to go back, and amplifying oscillations ensue.

Returning to thermally-driven systems as (probably) the easiest ones to visualize, we may estimate the thermal buoyancy force which a parcel experiences as it passes its equilibrium level z. We first linearize the heat conduction equation for a Boussinesq fluid [see (3.3)] about the motionless state with a uniform vertical temperature gradient $dT_0/dz < 0$. Then we seek separable solutions for the temperature field $T = \mathrm{Re}\,[\hat{T}(x, y, z)\, e^{-i\omega t}]$, with ω real, corresponding to the case of neutral stability. In this way we easily show that at the instant that any given parcel passes its equilibrium level from below, its temperature excess over that of the surroundings is

$$\Delta T = -h\frac{dT_0}{dz}\,\frac{\omega/\kappa s^2}{1+\omega^2/\kappa^2 s^4}, \tag{1.1}$$

where h is the amplitude of its vertical displacement, κ is the thermal diffusivity, and s is the total wavenumber of the disturbance, which satisfies $\nabla^2\hat{T}+s^2\hat{T} = 0$. On the downward crossing ΔT has the same magnitude, but the opposite sign. In each case the buoyancy force, which is a constant multiple of ΔT, is such as to promote an increase in amplitude of the oscillations. By hypothesis (of marginal stability) this is just balanced by stabilizing effects associated with the restoring forces, which we do not need to specify in detail for the purposes of the present argument.

The significant point is that ΔT has its maximum value, for a given h, when the frequency is equal to the thermal diffusion rate, i.e. when $\omega/\kappa s^2 = 1$. If $\omega/\kappa s^2$ is large, however, ΔT is much smaller than its maximum conceivable value, and this is because there is time during each downward/upward half-cycle for a parcel to acquire/lose only a rather small amount of heat. In the extreme case $\kappa = 0$, after all, $\Delta T = 0$ and the overstability mechanism fails altogether. If $\omega/\kappa s^2$ is *small*, on the other hand, ΔT is again much smaller than its maximum conceivable value, and the overstability mechanism is again somewhat ineffective. The reason is that thermal diffusion is fast enough for a parcel to acquire almost the same temperature as its local surroundings at every stage of the oscillation. When displaced downward it quickly heats up, but as it moves back up it quickly loses most of that heat again to its surroundings and arrives back with quite a small *net* temperature and density change.

Suppose, therefore, that we have a thermally driven system in which the above overstability mechanism exists, but that for some combination of parameters $\omega/\kappa s^2$ is small at marginal stability, so that the mechanism is comparatively ineffective and the critical adverse temperature gradient is correspondingly large. It is then comparatively easy to imagine how the addition of any restoring mechanism which increases the oscillation frequency *without* having any other significant side-effects might act as a catalyst for the overstability, by increasing the value of ΔT. In particular, one can envisage how by chemically stratifying the fluid in a 'bottom-heavy' way this increase in frequency might be achieved, and over-stability thus provoked. It would clearly be essential, however, for the solute diffusivity to be suitably small, because otherwise a downward-displaced parcel would pick up enough solute (by diffusion from the higher concentration in its surroundings) for the associated *increase* in density to outweigh any destabilizing effects of the enhanced heat transfer.

This is only a plausibility argument, of course, for even if ω is increased by some such means it does not automatically follow that $\omega/\kappa s^2$ will be increased, since the value of s at marginal stability, like that of ω itself, is not a quantity that can be externally imposed. Nevertheless, we present in § 2 evidence that all the recent† examples of 'stable' stratification acting as a catalyst for instability may be understood physically in these general terms, i.e. by identifying how the stratification affects the ratio of the oscillation frequency to the diffusion rate associated with the driving mechanism. These examples, for which appropriate references were given at the beginning of this paper, are all extremely complicated, and for this reason we present in § 3 a new example which has the advantage that it is sufficiently simple to permit a detailed and quantitative analysis of the physical processes involved. In this case at least the mechanism is just as described above. It was originally hoped that the system in § 3 would also constitute an example of the effect which could be realized in the laboratory, but despite its apparent similarity to the well-known thermo-solutal convection problem (Turner 1974) this is unfortunately not so. 'Stable' stratification has an anomalous effect in the system of § 3 only if the viscosity is little or no greater than the solute diffusivity, and the author knows of no fluid with this property. The purpose of including that particular example is therefore purely to lend detailed quantitative support to the general ideas advanced above.

2. Some specific instances of the mechanism at work?

Thermo-solutal convection in a rotating fluid

Pearlstein (1977, 1979) and Masuda (1978) have considered the effect of a 'bottom-heavy' solute gradient on the oscillatory instability of a plane fluid layer of depth d which is heated from below while rotating about a vertical axis with angular velocity Ω. They find that destabilization can occur, so that the critical Rayleigh number is lowered, if $\kappa > \nu > \kappa_s$, where ν denotes kinematic viscosity and κ_s denotes solute diffusivity. The effect seems most marked at fairly small values of $\sigma^2\mathscr{T}$ (more precisely, $\sigma^2\mathscr{T}/\pi^4$), where

$$\mathscr{T} \equiv 4\Omega^2 d^4/\nu^2 \tag{2.1}$$

is the Taylor number and $\sigma \equiv \nu/\kappa$ is the Prandtl number (see, for example, figure 4*b* of Masuda 1978).

When solute stratification is absent, one may infer from the results of Chandrasekhar (1961, chap. 2, § 39) that when $\sigma \lesssim 1$ and $\sigma^2\mathscr{T}$ is large the over-stable inertial oscillations have frequency close to the thermal diffusion rate, so that the over-stability mechanism is at the outset working very effectively, in the sense described in § 1. For sufficiently low values of $\sigma^2\mathscr{T}$, however, the frequency of the inertial oscillations which the Coriolis forces are able to support inevitably falls well below the thermal diffusion rate, and in such circumstances it is therefore understandable that a bottom heavy solute gradient should render the instability mechanism more effective by increasing the frequency at marginal stability. (Since κ_s is the smallest diffusivity, the stabilizing effect of the solute gradient via solute diffusion is, on the other hand,

† Our discussion in no way explains a whole class of rather earlier and very different examples involving plane parallel shear flows (see Howard & Maslowe 1973, Davey & Reid 1977 for discussion and further references). The key distinction is that diffusive effects do *not* play a crucial role in these other examples.

insignificant.) In order to test the applicability in this context of the ideas put forward in § 1, Pearlstein (private communication) has recently carried out calculations of $\omega/\kappa s^2$ at marginal stability, and it is notable that at the point at which the critical Rayleigh number is a minimum with respect to the solute stratification, this quantity is always fairly close to unity.

Magnetic instabilities of a rotating stratified fluid

The studies to be discussed here attempt, sometimes by means of rather simpler models, to infer the stability properties of a uniformly rotating fluid sphere under the influence of an azimuthal magnetic field B and a thermally-induced radial density distribution. We shall use a cylindrical polar coordinate system (r, θ, z) and for simplicity consider only the most well studied case in which B is proportional to r, the distance from the rotation axis. For geophysical and astrophysical applications, the Taylor and Hartmann numbers are very large, and the stability properties then depend critically on two key parameters

$$q \equiv \kappa/\eta \quad \text{and} \quad \mathscr{C} \equiv V^2/\Omega\eta, \tag{2.2}$$

where V is a representative Alfvén speed (proportional to the magnetic field), η is the magnetic diffusivity (proportional to the electrical resistivity) and Ω is the angular velocity. We shall take the radial density gradient to be bottom heavy, so that the only energy source for any instability is the magnetic field, and according to § 1 the key quantity for consideration is then $\omega/\eta s^2$.

Consider first the case in which q is *small*, as in the Earth's liquid core (where $q \sim 10^{-6}$), and take $\mathscr{C} \gtrsim 1$. Soward (1979) investigates a simplified model bounded instead by the planes $z = 0, d$, with gravity acting in the negative z direction, and finds that a sufficiently large 'stable' stratification leads to instability. The quantity $\omega/\eta s^2$ is small, $O(q)$, in the marginal state (see his § 3.3 and figure 2). One peculiar, but clearly understood, feature of his particular geometry is that stability is never regained, no matter how large the stratification. Acheson (1979*c*) chooses instead to bound his model by concentric cylinders, $r = r_1, r_2$, with gravity acting in the negative r direction. Instability occurs under very similar conditions to those obtained by Soward, but re-stabilization takes place for a density gradient about $\mathscr{C}/q$ times steeper than that which triggered the instability. Notably, $\omega/\eta s^2$ is then at least of order unity, in fact $O(\mathscr{C}^{\frac{1}{2}})$. In addition to a numerical attack on the spherical problem, Fearn (1979) considers a model bounded by both Soward's planes *and* Acheson's cylinders, with gravity in the negative z direction, and finds instability and eventual re-stabilization at values of the stratification quite different to those in Acheson's model. Nevertheless the ideas of § 1 once again seem to be appropriate, for the quantity $\omega/\eta s^2$ is small, $O(q)$, when stratification triggers instability [see Fearn's equation (B 11)] but $O(1)$ when re-stabilization takes place (see the end of Fearn's penultimate paragraph).

For astrophysical applications of the theory to radiative stellar interiors, q is *large*, and compressibility effects in the form of 'magnetic buoyancy' (see, e.g., Acheson 1979*a*) can be important. Acheson (1978*a*) investigates two models of this kind, one with radial boundaries at $r = r_1, r_2$, and radial gravity, and one wholly planar model with straight field lines, the latter being to link up with earlier work by Roberts & Stewartson (1977). Despite important differences between the results of the two models (cf. pp. 485, 487 of Acheson 1978*a*, and see also § 5.4 of Acheson 1979*a*), the point at which stratification stops destabilizing and starts re-stabilizing coincides in each case with the point at which $\omega/\eta s^2$ passes through $O(1)$ values [and in the plane model $\omega/\eta s^2$ is precisely unity at this point; see (7.32)–(7.34) of Acheson 1978*a*]. One poorly understood feature is that increases in stratification in these models produce consistent *decreases* in $\omega/\eta s^2$ at marginal stability, in contrast to one's normal expectations, so this quantity *falls* through $O(1)$ values at the change-over point. The main ideas of § 1 seem no less applicable, however, for this curious fact. Inside a certain 'critical radius' in Acheson's (1978*a*) cylindrical model (see also Acheson & Gibbons 1978*a*) the effects of magnetic buoyancy are comparatively unimportant and the dynamics closely resembles that of an incompressible fluid. There is thus a link between that work and the large q incompressible investigations of Roberts & Loper (1979), Soward (1979, § 3.2) and Fearn [1979, § 3 (*e*) and appendix B (*b*)], which makes the author optimistic that the ideas of § 1 may go some way toward explaining those results also, but without further investigation of those models, particularly regarding $\omega/\eta s^2$ and re-stabilization, this is somewhat speculative.

3. A simple illustrative example

The ideas put forward in § 1 in fact emanated from the following study of 'triply diffusive' convection, i.e. the natural extension of the well-known doubly diffusive convection problem (Turner 1974) to the case when an adverse temperature gradient contends with the presence of *two* bottom-heavy gradients of solute. Some interesting aspects of this problem have already been revealed by Griffiths (1979) and, in a very different (astrophysical) context (in which the same equations nevertheless apply, after some preliminary approximations), by Acheson & Gibbons (1978*b*). We are here concerned, however, with seeing if we can find, and then understand, the anomalous effect of density stratification.

The basic equations are, in the Boussinesq approximation:

$$\bar{\rho}\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{g} + \bar{\rho}\nu\nabla^2\mathbf{u}, \tag{3.1}$$

$$\nabla.\mathbf{u} = 0, \tag{3.2}$$

$$\frac{DT}{Dt} = \kappa\nabla^2 T, \tag{3.3}$$

$$\frac{DS_1}{Dt} = \kappa_1\nabla^2 S_1, \tag{3.4}$$

$$\frac{DS_2}{Dt} = \kappa_2\nabla^2 S_2, \tag{3.5}$$

$$\rho = \bar{\rho}(1-\alpha T+\beta_1 S_1+\beta_2 S_2). \tag{3.6}$$

Here ρ denotes density, $\mathbf{u} = (u, v, w)$ velocity, t time, p pressure, $\mathbf{g}$ acceleration due to gravity, ν kinematic viscosity, κ thermal diffusivity and T temperature. The density depends linearly on the temperature and solute concentrations S_1 and S_2, $\bar{\rho}$ denoting a reference density attained when all these are zero. Equations (3.4) and (3.5) describe the diffusion and advection of the solutes, and their diffusivities κ_1 and κ_2 have the same dimensions as ν and κ.

The basic state is one of no motion and hydrostatic balance,

$$\frac{dp}{dz} = -\rho g, \tag{3.7}$$

and we suppose that constant [so as to satisfy (3.3)–(3.5)] vertical gradients of temperature and solute are maintained between two horizontal boundaries at $z = 0$ and $z = d$. These are imagined to be rigid, stress-free and perfect conductors of heat and solute. By considering small-amplitude disturbances for which all perturbation quantities ϕ' can be written in the form

$$\phi' = \mathrm{Re}\,[\hat{\phi}(z)\exp i(kx-\omega t)], \tag{3.8}$$

where Re denotes the real part, we find that $\hat{w}(z) \propto \sin mz$, where m is an integral multiple of π/d, and arrive by the standard method at the following dispersion relationship:

$$\frac{D}{\omega+i\kappa s^2} = \frac{R_1}{\omega+i\kappa_1 s^2} + \frac{R_2}{\omega+i\kappa_2 s^2} - (\omega+i\nu s^2)\frac{s^2}{k^2}. \tag{3.9}$$

Here $s \equiv (k^2+m^2)^{\frac{1}{2}}$ denotes the wavenumber and

$$D \equiv -g\alpha\frac{dT}{dz}, \quad R_1 = -g\beta_1\frac{dS_1}{dz}, \quad R_2 = -g\beta_2\frac{dS_2}{dz}. \tag{3.10}$$

These quantities are all positive in the circumstances which we wish to investigate, and they all have dimension (frequency)^2; indeed $R_1^{\frac{1}{2}}$ and $R_2^{\frac{1}{2}}$ are the frequencies at which a displaced parcel would bob up and down under the influence of an individual solute gradient, all other processes, including diffusion, being neglected. The symbols in (3.10) are intended as an aid to the memory: D represents the driving mechanism for the instability, while R_1 and R_2 represent the two restoring agencies. We also introduce for convenience the parameters

$$\sigma_1 \equiv \kappa_1/\kappa, \quad \sigma_2 \equiv \kappa_2/\kappa. \qquad (3.11), (3.12)$$

While we have omitted the details of the analysis, one relationship between the perturbation variables will turn out to be of great value. If we consider the vertical displacement ξ' of a given parcel from its equilibrium position, so that $w' = \partial\xi'/\partial t$, then

$$\hat{\rho} = \frac{\omega\bar{\rho}\hat{\xi}}{g}\left[\frac{-D}{\omega+i\kappa s^2} + \frac{R_1}{\omega+i\kappa_1 s^2} + \frac{R_2}{\omega+i\kappa_2 s^2}\right]. \tag{3.13}$$

In the absence of diffusion, (3.9) predicts instability if and only if the net density increases with height. We shall suppose that this is not the case, taking

$$D < R_1 + R_2. \tag{3.14}$$

Indeed, we shall soon focus attention on situations in which D is very much less than R_1+R_2, instability being made possible only by a large disparity between the value of κ and the values of κ_1 and κ_2.

Let us now assume that the thermal diffusivity exceeds the solute diffusivities, so that $\kappa > \kappa_1$ and $\kappa > \kappa_2$. It is easy to show that any mode of instability is then oscillatory.

In the investigation of this overstability, the following procedure appears to be quite revealing. Rewrite (3.9) as

$$D = R_1\left(\frac{\omega+i\kappa s^2}{\omega+i\kappa_1 s^2}\right)+R_2\left(\frac{\omega+i\kappa s^2}{\omega+i\kappa_2 s^2}\right)-\frac{s^2}{k^2}[\omega^2-\nu\kappa s^4+i(\nu+\kappa)s^2\omega]. \tag{3.15}$$

The real part of this gives, at marginal stability (i.e. ω real),

$$D = R_1\left(\frac{\omega^2+\kappa\kappa_1 s^4}{\omega^2+\kappa_1^2 s^4}\right)+R_2\left(\frac{\omega^2+\kappa\kappa_2 s^4}{\omega^2+\kappa_2^2 s^4}\right)-(\omega^2-\kappa\nu s^4)\frac{s^2}{k^2}, \tag{3.16}$$

while the imaginary part gives

$$\frac{R_1(1-\sigma_1)}{\omega^2+\kappa_1^2 s^4}+\frac{R_2(1-\sigma_2)}{\omega^2+\kappa_2^2 s^4}-(1+\sigma)\frac{s^2}{k^2}=0. \tag{3.17}$$

Finally, multiply (3.17) by ω^2 and subtract the result from (3.16) to obtain

$$D = \left[\frac{\sigma_1 R_1}{\omega^2+\kappa_1^2 s^4}+\frac{\sigma_2 R_2}{\omega^2+\kappa_2^2 s^4}+\sigma\frac{s^2}{k^2}\right](\omega^2+\kappa^2 s^4). \tag{3.18}$$

We may regard (3.18) as an expression for the critical temperature gradient, measured by D, with ω being calculated from (3.17).

After some exploration we find the desired anomalous effect in the following (hypothetical) parameter regime:

$$R_1^{\frac{1}{2}}d^2 \sim R_2^{\frac{1}{2}}d^2 \sim \kappa \gg \kappa_1 \sim \kappa_2 \sim \nu. \tag{3.19}$$

From (3.17) we obtain to first approximation

$$\frac{R_1}{\omega^2+\kappa_1^2 s^4}+\frac{R_2}{\omega^2+\kappa_2^2 s^4}-\frac{s^2}{k^2}=0. \tag{3.20}$$

Inspection of (3.18) reveals the possibility of keeping its first two terms as small as about $\sigma_1 R_1$, but no smaller, since $\kappa \gg \kappa_1$. This can only be achieved, however, if $\omega \gg \kappa_1 s^2$, because otherwise both terms are larger by a factor of order κ^2/κ_1^2. Thus from (3.20) the frequency must be given to good approximation by

$$\omega^2 = (R_1+R_2)\,k^2/s^2, \tag{3.21}$$

and (3.18) then becomes

$$D = \left(\frac{\sigma_1 R_1+\sigma_2 R_2}{\omega^2}+\sigma\frac{s^2}{k^2}\right)(\omega^2+\kappa^2 s^4). \tag{3.22}$$

We can next use (3.21) to eliminate ω from (3.22), and we then minimize D with respect to k and m by choosing

$$k = \pi/(2^{\frac{1}{2}}d), \quad m = \pi/d. \tag{3.23}$$

We thus obtain

$$D = [(\sigma_1+\sigma)R_1+(\sigma_2+\sigma)R_2]\left[1+\frac{27\pi^4\kappa^2/4d^4}{R_1+R_2}\right]. \tag{3.24}$$

Alternatively, introducing the dimensionless parameters

$$(\mathscr{D},\mathscr{R}_1,\mathscr{R}_2) \equiv \frac{4d^4}{27\pi^4\kappa^2}(D,R_1,R_2), \tag{3.25}$$

we may write (3.24) as

$$\mathscr{D} = (\sigma_1+\sigma)(\mathscr{R}_1+\alpha\mathscr{R}_2)[1+(\mathscr{R}_1+\mathscr{R}_2)^{-1}]. \tag{3.26}$$

It is evident from (3.26) that, as we increase $\mathscr{R}_2$ from zero, $\mathscr{D}$ will initially decrease, so that instability is facilitated, if the parameter

$$\alpha \equiv (\sigma_2+\sigma)/(\sigma_1+\sigma) \tag{3.27}$$

is sufficiently small. By differentiation we easily obtain the following precise condition for this to happen:

$$\alpha < (1+\mathscr{R}_1)^{-1}. \tag{3.28}$$

The effect of $\mathscr{R}_2$ on the stability boundaries is illustrated in figure 1, where $\mathscr{D}$ is regarded as fixed. When $\mathscr{R}_2 = 0$ instability occurs for the system in figure 1 (*a*) if $\mathscr{R}_1$ is less than 0·2, while figure 1 (*b*) is for a system with a rather larger $\mathscr{D}$ and instability occurs when $\mathscr{R}_1$ drops below unity, again if $\mathscr{R}_2 = 0$. The critical values of $\mathscr{R}_1$, given $\mathscr{D}$, are plotted (horizontally) in each case against $\mathscr{R}_2$ for four different values of α, and there is instability to the left of such curves and stability to the right. Looking at the curve for $\alpha = 0{\cdot}2$ in figure 1(*a*), for example, we see how any originally stable system with $\mathscr{R}_1$ greater than 0·2 but less than about 0·53 is initially rendered unstable when $\mathscr{R}_2$ reaches some value less than unity, but then becomes stable again as $\mathscr{R}_2$ is increased further. It is evident from these curves that with smaller values of α a still larger value of $\mathscr{R}_2$ is needed before the second solute begins to play a stabilizing role, and this may be seen by inspection of (3.26). On comparing figures 1 (*a*) and 1 (*b*) it is also clear that, given α, the anomalous effect is more pronounced when the critical value of $\mathscr{R}_1$ in the absence of the second solute is small, i.e. when the frequency of oscillation associated with the first solute gradient is less than the thermal diffusion rate. This is of course in keeping with the ideas put forward in § 1.

The present example, however, is so simple that we can investigate the precise nature of the physical mechanism more deeply. In the regime (3.19) we have $\omega \gg \kappa_1 s^2 \sim \kappa_2 s^2$, and we also have, from (3.24), that $D \sim \sigma_1 R_1 \sim \sigma_2 R_2$. Since σ_1 and σ_2 are small we may therefore approximate (3.13) by

$$\hat{\rho} = \bar{\rho}\hat{\xi}(R_1+R_2+i\delta)/g \tag{3.29}$$

where

$$\delta \equiv \frac{\kappa s^2}{\omega}\left(\frac{D}{1+\kappa^2 s^4/\omega^2}-\sigma_1 R_1-\sigma_2 R_2\right). \tag{3.30}$$

Notably, if viscosity were absent (3.22), which leads directly to the marginal stability equation (3.24), could be written $\delta = 0$. Putting the matter another way, if D exceeds D_c, the critical value for instability, then comparing (3.22) with (3.30) we see that $\delta > 0$, and bearing in mind (3.8), this means that when $D > D_c$ the actual (real) perturbation density ρ' at any fixed point oscillates in such a way as to lag slightly behind the displacement field ξ'.

There are three separate contributions to this discrepancy between the parcel density and the ambient density as it passes its equilibrium position, of course, and the three terms in (3.30) represent the density changes which a parcel experiences by exchanging heat, first solute and second solute respectively with its surroundings. When below its equilibrium position it loses density by heating up, but it also inevitably acquires some solute by diffusion, and the first effect must outweigh the second if the parcel is to be 'light' on return to its equilibrium position. We have assumed that κ_1 and κ_2 are small, in the sense that the solute diffusion rates are much less than the oscillation frequency. As a result, the density change due to diffusion of each solute is small and simply

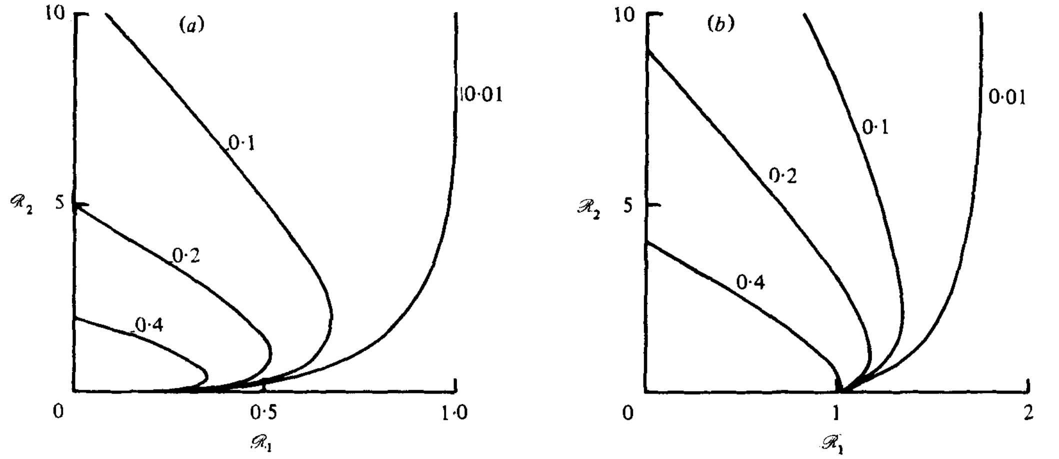

FIGURE 1. Two stability diagrams for different prescribed adverse temperature gradients, each plotted for four different (labelled) values of $\alpha = (\kappa_2+\nu)/(\kappa_1+\nu) < 1$. The parameters $\mathscr{R}_1$ and $\mathscr{R}_2$ act as measures of the two bottom-heavy solute gradients (see text). Instability occurs to the left of each curve and is at first *promoted* by increasing $\mathscr{R}_2$.

proportional to the solute diffusivity and (via R_1 or R_2) the concentration gradient, for obvious reasons. The density change resulting from heat diffusion has a more complicated form, however, as a consequence of our assumption in (3.19) that κ is quite large, and the dependence of the first term in (3.30) on the key quantity $\kappa s^2/\omega$ is *not* monotonic. It has a maximum possible value of $\frac{1}{2}D$ when $\omega = \kappa s^2$, but the actual value is a good deal smaller than this if the oscillation frequency is either much smaller or much larger than the thermal diffusion rate, as argued in § 1.

Ignoring viscous effects entirely for a moment, let us suppose that the system is stable when $R_2 = 0$. As we have remarked, this would imply that $\delta \leqslant 0$ when $R_2 = 0$. Inspection of (3.30) and (3.21) reveals quite clearly that if R_2 is increased from zero it has two effects. It increases ω, and unless the original value of ω greatly exceeded κs^2 [which our hypothesis (3.19) effectively excludes] this leads to a significant increase in the thermal term $D/(1+\kappa^2 s^4/\omega^2)$ in (3.30), which promotes instability. On the other hand it also introduces a negative term $-\sigma_2 R_2$ which represents additional density acquired (lost) by the parcel due to *second* solute diffusion on the downward (upward) journey, and this is a stabilizing effect. With only one solute present the latter effect always 'wins', and increasing R_1 with $R_2 = 0$ simply acts as a stabilizing influence, as may be checked from (3.24). It is evidently reasonable, however, that if κ_2/κ_1 is sufficiently small, increasing the bottom-heavy gradient of a *second* solute need make no significant difference to the (stabilizing) amount of solute acquired by a parcel by diffusion, while at the same time it may increase the oscillation frequency to a value which brings the *heat* exchange between a parcel and its surroundings closer to that needed for the optimum performance of the over-stability mechanism.

When viscous effects are present, however, δ has to exceed some non-zero value for instability, in fact given by (3.22):

$$\delta > \nu\omega s^4/k^2. \tag{3.31}$$

Comparing this with (3.30) we see that if, as in reality, ν greatly exceeds the solute diffusivities κ_1 and κ_2, the viscous drag forces on the parcel are so much more significant than the buoyancy force fluctuations wrought by solute diffusion as a stabilizing mechanism that no amount of juggling with these diffusivities can achieve the desired end; α is almost unity if $\nu \gg \kappa_1, \kappa_2$.

4. Concluding remarks

We have attempted to explain in simple physical terms a number of recent examples in which linear theory predicts that 'stable' density stratification acts as a catalyst for instability. In all cases (multiply) diffusive effects were crucial to the argument, which applies only to instabilities of oscillatory type, i.e. 'over-stability'. The unifying idea was that the over-stability mechanism, whether thermal or otherwise, works most effectively when the oscillation frequency and the diffusion rate associated with the driving mechanism are closely matched at marginal stability. 'Stable' density stratification can destabilize the system by shifting the oscillation frequency so as to produce a better match.

While we have focused attention on the anomalous effect of bottom-heavy density gradients, the ideas may in principle apply to other systems in which an ingredient normally found to be a restoring or stabilizing force has the opposite effect. One such example is the way in which the thermal instability of a rotating fluid layer may be facilitated by the addition of a uniform vertical magnetic field. The usual explanation for this (see, e.g. Acheson 1978*b*) is satisfactory when the instability occurs as steady convection, but Acheson (1979*b*) has recently argued that when the instability occurs in an oscillatory manner at high values of q the mechanism responsible is instead the one discussed in this paper. Certainly, when the critical Rayleigh number is plotted against the field strength the point at which the curve has a minimum coincides with that at which $\omega/\kappa s^2 = 1$ in the marginally stable state.

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